Midterm Solutions

1. Let $f(x,y) = \begin{cases} 1, & y \geq x^4 \\ 1, & y \leq 0 \\ 0, & \text{otherwise.} \end{cases}$

   Find each of the following limits, or explain that the limit does not exist.

   (a) $\lim_{(x,y)\to(0,1)} f(x,y)$

   (b) $\lim_{(x,y)\to(2,3)} f(x,y)$

   (c) $\lim_{(x,y)\to(0,0)} f(x,y)$

   Solution. (a) Any point (x,y) inside a ball of sufficiently small radius (say $r < 0.5$) around $(0,1)$, satisfies $y \geq x^4$. From this, we can infer that

$$\lim_{(x,y)\to(0,1)} f(x,y) = 1.$$

   (b) It is easy to see that any point (x,y) inside a ball of sufficiently small radius (say $r < 0.5$) around $(2,3)$, does not satify either $y \geq x^4$ or $y \leq 0$. Hence,

$$\lim_{(x,y)\to(2,3)} f(x,y) = 0.$$

   (c) Along $x = 0$, we can see that

$$\lim_{h\to 0} f(0,h) = f(0,-h) = 1,$$

   which implies that

$$\lim_{(x,y)\to(0,0)} f(x,y) = 1.$$

   However, along the curve $y = x^2$, both $y \geq x^4$ and $y \leq 0$ are not satisfied, which wound imply that

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0.$$

   Hence, the limit does not exist.

2. Let $f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$

   Show that $f_x(0,0)$ and $f_y(0,0)$ exist, but f is not differentiable at the origin.

   Solution. By definition, we know that

$$f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \frac{0}{h} = 0.$$

   In a similar manner, we can also show that $f_y(0,0) = 0$.

   Moreover, along the curve $x = y^2$, we can see that

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{y^4}{y^4+y^4} = \frac{1}{2}.$$

   But by definition, $f(0,0) = 0$, which shows that f is not continuous at the origin. Hence, f cannot be differentiable at the origin.

3. If $G = f(x,y)$, where $x = r\cos\theta$ and $y = r\sin\theta$. Them show that

$$(G_x)^2 + (G_y)^2 = (G_r)^2 + \frac{(G_\theta)^2}{r^2}.$$

   Solution. By Chain Rule,

$$G_r = f_x x_r + f_y y_r = G_x(\cos\theta) + G_y(\sin\theta)$$

   and

$$G_\theta = f_x x_\theta + f_y y_\theta = G_x(-r\sin\theta) + G_y(r\cos\theta).$$

   From this, it is apparent that the equation above holds good.

4. Consider the scalar field $f(x,y) = x^2 + kxy + y^2$.

   (a) Show that $(0,0)$ is a critical point for $f(x,y)$ for any $k \in \mathbb{R}$.

   (b) For what values of k will $f(x,y)$ have a saddle point at $(0,0)$.

   (c) For what values of k does f have local minimum at $(0,0)$?

   (d) For what values of k is the Second Derivative Test inconclusive?

Solution. (a) When $k = 0$, $f_x(x, y) = 2x$ and $f_y(x, y) = 2y$. Equating f_x and f_y to zero, we get that $(0, 0)$ is a critical point.

If $k \neq 0$, then

$$f_x(x, y) = 0 \implies 2x + ky = 0 \implies y = -\frac{2}{k}x.$$

Upon substituting this in $f_y(x, y) = 0$, we have

$$kx + 2(-\frac{2}{k}x) = 0 \implies (k - \frac{4}{k})x = 0 \implies x = 0 \text{ or } k = \pm 2.$$

If $x = 0$, then $y = 0$, and if $k = \pm 2$, then $y = \pm x$, In both cases, $(0, 0)$ is a critical point.

(b) The second partial derivatives are: $f_{xx} = f_{yy} = 2$, and $f_{xy} = f_{yx} = k$. From these, we obtain $\Delta(x, y) = f_{xx}f_{yy} - {f_{xy}}^2 = 4 - k^2$. For f to have a saddle point at $(0, 0)$, we need $\Delta(0, 0) < 0$, which would imply that $4 < k^2$, or $k \in (-\infty, 2) \sqcup (2, \infty)$.

(c) f will have a local minimum at $(0, 0)$ when $\Delta(0, 0) > 0$, that is, when $4 - k^2 > 0$, or $k \in (-2, 2)$.

(d) The test is inconclusive when $f_{xx} = 4 - k^2 = 0$, that is when $k = \pm 2$.

5. Show that the sum of the x-, y-, and z-intercepts of any tangent plane to the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{c}$ is a constant. Also, find this constant. (Note that the *x-intercept* of a graph is the x-coordinate of the point where it meets the x-axis. The y- and z-intercepts are defined analogously.)

   Solution. Let $P(x_0, y_0, z_0)$ be any point on the surface. Then the equation of the tangent plane T_P at P can be shown to be

$$\frac{x}{\sqrt{x_0}} + \frac{y}{\sqrt{y_0}} + \frac{z}{\sqrt{z_0}} = \sqrt{c}.$$

   Clearly, the x-, y-, and z-intercepts of T_P are $\sqrt{cx_0}$, $\sqrt{cy_0}$, and $\sqrt{cz_0}$ respectively. The sum of these intercepts is $\sqrt{c}(\sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0}) = c$, which is a constant.

6. Find the point closest to the origin on the curve of intersection of the plane $2y + 4z = 5$ and the cone $z^2 = 4x^2 + y^2$.

   Solution. Let (x, yz) be an arbitrary point on the curve of intersection. Let $f(x, y, z) = x^2 + y^2 + z^2$ be the square of its distance from the origin. We want to minimize f subject to the constraints $g(x, y, z) = 2y + 4z$ and $h(x, y, z) = 4x^2 + y^2 - z^2$.

By the method of Lagrange's Multipliers, we need to simultaneously solve the following system of equations

$$\begin{aligned} \nabla f &= \lambda \nabla g + \mu \nabla h \\ 2y + 4z &= 5 \\ z^2 &= 4x^2 + y^2. \end{aligned}$$

This yields the following system of five equations in five unknowns,

$$\begin{aligned} 2x &= 8x\mu \\ 2y &= 2\lambda + 8y\mu \\ 2z &= 4\lambda - 2z\mu \\ 2y + 4z &= 5 \\ z^2 &= 4x^2 + y^2. \end{aligned}$$

There are two possible solutions to the first equation in the system: $x = 0$ (Case 1) or $\mu = \frac{1}{4}$ (Case 2). We now examine these two cases.

Case 1: If $x = 0$, then from the fifth equation we get that $z = \pm y$. Upon substituting this in the fourth equation, we get that $y = \frac{5}{6}$ or $y = -\frac{5}{2}$. Therefore, the solutions we obtain from this case are $(0, \frac{5}{6}, \frac{5}{6})$ and $(0, -\frac{5}{2}, \frac{5}{2})$.

Case 2: If $\mu = \frac{1}{4}$, then from the second equation, we can infer that $\lambda = 0$, and by substituting this in the third equation, we get $z = 0$. From the fourth equation, we have that $y = \frac{5}{2}$, but this cannot be a feasible solution, as this would then make it impossible to find an x that satisfies the fifth equation.

It is easy to see that among the feasible points, $(0, \frac{5}{6}, \frac{5}{6})$ is closest to origin.

7. Find the absolute maximum and minimum values of $f(x, y) = 3 + xy - x - 2y$ on the closed triangular region with vertices $(1, 0)$, $(5, 0)$, and $(1, 4)$.

   Solution. Let D denote the closed traingular region with the three vertices $(1, 0)$, $(5, 0)$, and $(1, 4)$. Since f is a polynomial, it is continuous in D, and hence attains its extremal values in D (by the Extreme Value Theorem). Setting $f_x = f_y = 0$, we obtain $(2, 1)$ as the only critical point, and $f(2, 1) = 1$.

Note that ∂D is a triangle comprised of three line segments: A segment L_1 joining $(1,4)$ and $(1,0)$, a segment L_2 joining $(1,0)$ and $(5,0)$, and a segment L_3 joining $(5,0)$ and $(1,4)$. We will now compute the extremal values of f along ∂D.

Along L_1: $x = 1$ and $f(1, y) = 2 - y$ for $y \in [0, 4]$, which is a decreasing function in y. So the maximum value is $f(1, 0) = 2$ and the minimum value is $f(1, 4) = -2$.

Along L_2: $y = 0$ and $f(x, 0) = 3 - x$ for $x \in [1, 5]$, which is a decreasing function in x. Hence, the maximum value is $f(1, 0) = 2$ and the minimum value is $f(5, 0) = -2$.

Along L_3: $y = 5 - x$ and $g(x) = f(x, 5 - x) = -(x - 3)^2 + 2$ for $x \in [1, 5]$. This function has a maximum at $x = 3$ (which we can conclude by setting $g'(x) = 0$), where $f(3, 2) = 2$, and a minimum at both $(1, 4)$ and $(5, 0)$, where $f(1, 0) = f(5, 4) = -1$.

Therefore, the absolute maximum of f on D is $f(1, 0) = f(3, 2) = 2$, and the absolute minimum is $f(1, 4) = f(5, 0) = -2$.

8. **(Bonus)** Using Lagrange's Multipliers, deduce that if $x_1, x_2, \ldots, x_n$ are positive numbers, then

$$\sqrt[n]{x_1 x_2 \ldots x_n} \leq \frac{1}{n} \sum_{i=1}^{n} x_i.$$

In other words, this inequality says that the geometric mean of n numbers is no larger than the arithmetic mean of the numbers.

Solution. We wish to maximize $f(x_1, \ldots, x_n) = \sqrt[n]{x_1 \ldots x_n}$ subject to the constraint $g(x_1, \ldots, x_n) = x_1 + \ldots + x_n = c$ and $x_i > 0$. Then

$$\nabla f = \left(\frac{1}{n}(x_1 \ldots x_n)^{\frac{1}{n}-1}(x_2 \ldots x_n), \ldots, \frac{1}{n}(x_1 \ldots x_n)^{\frac{1}{n}-1}(x_1 \ldots x_{n-1}) \right),$$

and $\nabla g = (1, \ldots, 1)$. Since the x_i can be chosen so that f can be made arbitraily close to zero, Y cannot be a point where f attains its minimum. Thus Y is a point of maximum and hence $f(X) \leq f(Y)$, for all $X = (x_1, \ldots, x_n) \in \mathbb{R}^n$. In other words, we have the inequality

$$\sqrt[n]{x_1 \ldots x_n} \leq \sqrt[n]{\frac{c}{n} \ldots \frac{c}{n}} = \frac{c}{n}.$$

Equating $\nabla f = \lambda \nabla g$ and using Lagrange's Multipliers, we have the following system of $n+1$ equations

$$\frac{1}{n}(x_1 \ldots x_n)^{\frac{1}{n}-1}(x_2 \ldots x_n) = \lambda \implies (x_1 \ldots x_n)^{1/n} = n\lambda x_1$$
$$\vdots$$
$$\frac{1}{n}(x_1 \ldots x_n)^{\frac{1}{n}-1}(x_1 \ldots x_{n-1}) = \lambda \implies (x_1 \ldots x_n)^{1/n} = n\lambda x_n$$
$$x_1 + \ldots + x_n = c$$

Thus we have that $\lambda \neq 0$ and $x_1 = x_2 = \ldots = x_n$. (For if $\lambda = 0$, then we cannot have that all $x_i > 0$.) Then the last equation in the system would imply that $x_i = \frac{c}{n}$, for $1 \leq i \leq n$. We conclude that the only point (or vector) where f will have an extremal value is $Y = \left(\frac{c}{n}, \ldots, \frac{c}{n}\right)$.

Since the x_i can be chosen so that f can be made arbritraily close to zero, Y cannot be a point where f attains its minimum. Thus Y is a point of maximum and hence $f(X) \leq f(Y)$, for all $X = (x_1, \ldots, x_n) \in \mathbb{R}^n$. In other words, we have the inequality

$$\sqrt[n]{x_1 \ldots x_n} \leq \sqrt[n]{\frac{c}{n} \ldots \frac{c}{n}} = \frac{c}{n}.$$

Since $x_1 + \ldots + x_n = c$, we have the required result.